LINEAR SPACES ON CUBIC HYPERSURFACES, AND PAIRS OF HOMOGENEOUS CUBIC EQUATIONS

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1. Introduction

A remarkable theorem of Birch [2] shows that a system of homogeneous polynomials with rational coefficients has a non-trivial zero, provided only that these polynomials are of odd degree, and the system has sufficiently many variables in terms of the number and degrees of these polynomials. Despite four decades of effort, the problem of obtaining a reasonable bound for the latter number of variables has proved to be one of great difficulty. When the system consists of a single cubic form, Davenport [4] has succeeded in showing that 16 variables suffice, and Schmidt [17, 18, 19, 20] has devoted a series of papers to systems of cubic forms, showing in particular that 5140 variables suffice for pairs of cubic forms, and that $(10r)^5$ variables suffice for systems of r cubic forms. The current state of knowledge for forms of higher degree is, by comparison, extremely weak (but see [21, 22]), and so it seems worthwhile expending further effort on the case of systems of cubic forms. In this paper we improve on Schmidt's result for pairs of cubic forms. In contrast with the sophisticated versions of the Hardy-Littlewood method employed by Davenport and Schmidt, our approach is based on an elementary idea of Lewis [12], and is applicable in arbitrary number fields. This method also has consequences for the existence of linear spaces of rational solutions on cubic hypersurfaces, thereby improving on work of Lewis and Schulze-Pillot [13] on this topic.

Before describing our main theorem we require some notation. When K is a field, and r and m are non-negative integers, let $\gamma_K(r;m)$ denote the least integer (if any such integer exists) with the property that whenever $s > \gamma_K(r;m)$, and $f_i(\mathbf{x}) \in K[x_1, \dots, x_s]$ $(1 \le i \le r)$ are cubic forms, then the system of equations $f_i(\mathbf{x}) = 0$ $(1 \le i \le r)$ possesses a solution set which contains a linear subspace of K^s with projective dimension m. If no such integer exists, define $\gamma_K(r;m)$ to be $+\infty$. Also, let $\beta_K(r;m)$ denote the corresponding integer when the cubic forms are replaced by quadratic forms. We abbreviate $\gamma_K(r;0)$ to $\gamma_K(r)$, and $\gamma_K(1;0)$ to γ_K.

Theorem 1. *Let K be a field, let $d \in K$, and suppose that $\sqrt{d} \notin K$. Then on writing $L = K(\sqrt{d})$, one has $\gamma_K(1;m) \le \beta_L(m;\gamma_L) + \frac{1}{2}m(m+3)$. In particular,*

$$\gamma_K(1;m) \le m + (m+1)\gamma_L + \tfrac{1}{2}m(m+1)\left(\beta_L(1;0)+1\right).$$

We remark that when γ_L and $\beta_L(1;0)$ are both finite, Theorem 1 shows that $\gamma_K(1;m) \ll (m+1)^2$. For comparison, Lewis and Schulze-Pillot [13, Theorem 2] have shown that when γ_K is finite, one has

$$\gamma_K(1;m) \ll (m+1)^{(5+\sqrt{17})/2}. \tag{1}$$

The superiority of our new bound is transparent when the field K is a number field.

Theorem 2. *(a) Let p be a rational prime, and suppose that F is an algebraic extension of $\mathbb{Q}_p$ (possibly $\mathbb{Q}_p$ itself). Define δ_m to be 18 or 22 according to whether m is even or odd. Then for each non-negative integer m, one has*

$$\gamma_F(1;m) \le \tfrac{1}{2}(5m^2 + 21m + \delta_m).$$

(b) Let L be an algebraic extension of $\mathbb{Q}$ (possibly $\mathbb{Q}$ itself). Then for each non-negative integer m, one has

$$\gamma_L(1;m) \le \tfrac{1}{2}(5m^2 + 37m + 30).$$

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The bound of Theorem 2(a) improves on Lewis and Schulze-Pillot [13, Proposition 2][1], which yields $\gamma_{\mathbb{Q}_p}(1;m) \le 3m^2 + 12m + \varepsilon_m$, where ε_m is 9 when $m = 0$ or 1, and is 5 otherwise. Meanwhile the conclusion of Theorem 2(b) is a sizeable improvement on the sharpest bound available hitherto, namely that provided by the inequality (1) due to Lewis and Schulze-Pillot.

The bounds provided by Theorem 2 are of sufficient strength to yield immediate consequences concerning the solubility of pairs of cubic equations.

Corollary 1. *(a) Let p be a rational prime, and suppose that F is an algebraic extension of $\mathbb{Q}_p$ (possibly $\mathbb{Q}_p$ itself). Then $\gamma_F(2) \le 308$.*
(b) Let L be an algebraic extension of $\mathbb{Q}$ (possibly $\mathbb{Q}$ itself). Then $\gamma_L(2) \le 855$.

For comparison, Leep and Schmidt [9, equation (3.16p)] provide the bound $\gamma_{\mathbb{Q}_p}(2) \le 320$, and Schmidt [20, Theorem 1] has established the estimate $\gamma_{\mathbb{Q}}(2) \le 5139$. Thus, in addition to holding in arbitrary number fields, our bounds have the benefit of being sharper. We note, however, that both estimates fall far short of the conjectured bounds $\gamma_{\mathbb{Q}_p}(2) \le 18$ and $\gamma_{\mathbb{Q}}(2) \le 18$, which of course would be best possible. In this context we remark that the inequality $\gamma_{\mathbb{Q}_p}(2) \le 18$ is known to hold for all but a finite set of primes p, from work of Ax and Kochen [1], but no upper bound is currently known for the size of the possible exceptional primes.

It seems opportune to remark that by combining the conclusion of Corollary 1(b) to Theorem 2 with [20, equation (1.10)], one readily deduces the following modest improvement on Schmidt [20, Theorem 1].

Corollary 2. *When $t \in \mathbb{N}$ write $\widetilde{\gamma}(t) = \sup_p \gamma_{\mathbb{Q}_p}(t)$. Then when $r \ge 3$,*

$$\gamma_{\mathbb{Q}}(r) \le 855 + \sum_{t=3}^{r} \left(8t\widetilde{\gamma}(t) + 2t^2 - 2t \right).$$

This corollary provides the same conclusion as [20, Theorem 1], save that we have replaced 5139 in the latter by 855 in the former.

We establish Theorem 1 in §2 by using ideas of Leep and Schmidt [9] to deduce the existence of linear spaces of solutions over $K(\sqrt{d})$. Then, by exploiting an old idea of Lewis [12], we are able to pull linear spaces of solutions back from $K(\sqrt{d})$ to K. We provide proofs of Theorem 2 and its corollaries in §3, these results following from Theorem 1 in routine manner.

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2. Pulling back points from a quadratic extension

We begin this section by showing how linear spaces of K-rational solutions of a given cubic form can be obtained from corresponding linear spaces of $K(\sqrt{d})$-rational solutions. The key ingredients of our argument may be found in the proof of [12, Theorem II], where the case $d = -1$ is investigated. Since, in our application, we are concerned with keeping tight control of the number of variables arising, and moreover our conclusion is a little more general than that of Lewis, we provide an essentially complete exposition. We first describe some notation useful in handling cubic forms. Let K be a field, and suppose that $f(\mathbf{x}) \in K[x_1, \dots, x_s]$ is a cubic form. Then, for suitable coefficients $c_{ijk} \in K$, we can write $f(\mathbf{x})$ in the shape

$$f(\mathbf{x}) = \sum_{1 \le i \le j \le k \le s} c_{ijk} x_i x_j x_k,$$

and define the trilinear form $T(\mathbf{x}, \mathbf{y}, \mathbf{z})$ associated with f by

$$T(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{1 \le i \le j \le k \le s} c_{ijk} x_i y_j z_k.$$

It is convenient to define the polar forms f_{21}, f_{12} and f_{111} associated with f by

$$f_{21}(\mathbf{x}, \mathbf{y}) = T(\mathbf{x}, \mathbf{x}, \mathbf{y}) + T(\mathbf{x}, \mathbf{y}, \mathbf{x}) + T(\mathbf{y}, \mathbf{x}, \mathbf{x}), \quad f_{12}(\mathbf{x}, \mathbf{y}) = f_{21}(\mathbf{y}, \mathbf{x}),$$

[1] Here we have corrected two minor oversights in the proof of [13, Proposition 2]. The first arises in the case $m = 1$ through the use of the second inequality of [13, Proposition 1] when $r = 1$, in which case it is not valid. The second oversight occurs on lines 12 and 14 of page 279, where n should be replaced by $n - m$, since there are but $n - m$ variables $y_{m+1}, \dots, y_n$ in the forms to be solved.

$$f_{111}(\mathbf{x},\mathbf{y},\mathbf{z}) = T(\mathbf{x},\mathbf{y},\mathbf{z}) + T(\mathbf{x},\mathbf{z},\mathbf{y}) + T(\mathbf{y},\mathbf{z},\mathbf{x}) + T(\mathbf{y},\mathbf{x},\mathbf{z}) + T(\mathbf{z},\mathbf{x},\mathbf{y}) + T(\mathbf{z},\mathbf{y},\mathbf{x}).$$

Notice that $f_{21}(\mathbf{x},\mathbf{y})$ is homogeneous of degree 2 with respect to $\mathbf{x}$, and of degree 1 with respect to $\mathbf{y}$, and an analogous property holds for $f_{12}(\mathbf{x},\mathbf{y})$. Also $f_{111}(\mathbf{x},\mathbf{y},\mathbf{z})$ is symmetric and homogeneous of degree 1 with respect to each of its arguments.

Our proof of Lemma 2.1 below is essentially the same as that of [12, Lemma D] (see [14, page 109] for a geometric proof).

Lemma 2.1. *Let K be a field, let $d \in K$, and suppose that $\sqrt{d} \notin K$. If a cubic form $f(\mathbf{x}) \in K[\mathbf{x}]$ has a non-trivial zero over $K(\sqrt{d})$, then it has one over K.*

Proof. Suppose that $\boldsymbol{\gamma}$ is a non-trivial zero of $f(x_1,\ldots,x_s)$ over $K(\sqrt{d})$. We may write $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}\sqrt{d}$ with $\boldsymbol{\alpha},\boldsymbol{\beta} \in K^s$. Notice that we may suppose $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ to be linearly independent over K^s, for otherwise one or other of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is a non-trivial K-rational root of $f(\mathbf{x})$. Further, on substituting this expression for $\boldsymbol{\gamma}$ into the equation $f(\boldsymbol{\gamma}) = 0$, we obtain

$$f(\boldsymbol{\gamma}) = f(\boldsymbol{\alpha}) + \sqrt{d} f_{21}(\boldsymbol{\alpha},\boldsymbol{\beta}) + d f_{12}(\boldsymbol{\alpha},\boldsymbol{\beta}) + d\sqrt{d} f(\boldsymbol{\beta}) = 0,$$

whence, on considering components, $f(\boldsymbol{\alpha}) = -df_{12}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and $f_{21}(\boldsymbol{\alpha},\boldsymbol{\beta}) = -df(\boldsymbol{\beta})$. It may now be verified by substitution that $\boldsymbol{\alpha} f(\boldsymbol{\beta}) - \boldsymbol{\beta} f_{12}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is a K-rational zero of $f(\mathbf{x})$. Moreover, by the linear independence of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the latter zero will be non-trivial provided only that $f(\boldsymbol{\alpha})$ and $f(\boldsymbol{\beta})$ are not both zero. However, if $f(\boldsymbol{\alpha}) = f(\boldsymbol{\beta}) = 0$, we once again have a non-trivial K-rational zero of $f(\mathbf{x})$, and so the proof of the lemma is complete.

Equipped with Lemma 2.1, we now show how to pull linear spaces of solutions back from $K(\sqrt{d})$ to K.

Lemma 2.2. *Let K be a field, let $d \in K$, and suppose that $\sqrt{d} \notin K$. Suppose that a cubic form $f(\mathbf{x}) \in K[x_1,\ldots x_s]$ possesses linearly independent zeros $\mathbf{v}_1,\ldots,\mathbf{v}_n \in K^s$ with the property that for each $t_1,\ldots,t_n$ one has*

$$f(t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n) = 0. \tag{2}$$

If the system of equations

$$f(\mathbf{x}) = f_{12}(\mathbf{v}_i,\mathbf{x}) = f_{21}(\mathbf{v}_i,\mathbf{x}) = f_{111}(\mathbf{v}_i,\mathbf{v}_j,\mathbf{x}) = 0 \quad (1 \le i,j \le n), \tag{3}$$

has a solution over $K(\sqrt{d})$ which is linearly independent of $\mathbf{v}_1,\ldots,\mathbf{v}_n$ over $K(\sqrt{d})$, then $f(\mathbf{x})$ possesses linearly independent zeros $\mathbf{w}_0,\ldots,\mathbf{w}_n \in K^s$ with the property that for each $y_0,\ldots,y_n$ one has

$$f(y_0\mathbf{w}_0 + \cdots + y_n\mathbf{w}_n) = 0. \tag{4}$$

Proof. Suppose that $\boldsymbol{\gamma} \in K(\sqrt{d})^s$ is a solution of the system (3) which is linearly independent of $\mathbf{v}_1,\ldots,\mathbf{v}_n$ over $K(\sqrt{d})$. Then f vanishes on the $K(\sqrt{d})$-space spanned by $\boldsymbol{\gamma},\mathbf{v}_1,\ldots,\mathbf{v}_n$. We may write $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}\sqrt{d}$ with $\boldsymbol{\alpha},\boldsymbol{\beta} \in K^s$. On substituting $\mathbf{x} = \boldsymbol{\gamma}$ into the system (3), and considering components, we obtain

$$f_{21}(\mathbf{v}_i,\boldsymbol{\alpha}) = f_{21}(\mathbf{v}_i,\boldsymbol{\beta}) = f_{111}(\mathbf{v}_i,\boldsymbol{\alpha},\boldsymbol{\beta}) = 0 \quad (1 \le i \le n), \tag{5}$$

$$f_{111}(\mathbf{v}_i,\mathbf{v}_j,\boldsymbol{\alpha}) = f_{111}(\mathbf{v}_i,\mathbf{v}_j,\boldsymbol{\beta}) = 0 \quad (1 \le i,j \le n), \tag{6}$$

$$f_{12}(\mathbf{v}_i,\boldsymbol{\alpha}) = -df_{12}(\mathbf{v}_i,\boldsymbol{\beta}) \quad (1 \le i \le n), \tag{7}$$

and

$$f(\boldsymbol{\alpha}) = -df_{12}(\boldsymbol{\alpha},\boldsymbol{\beta}), \quad f_{21}(\boldsymbol{\alpha},\boldsymbol{\beta}) = -df(\boldsymbol{\beta}). \tag{8}$$

Furthermore, in view of the equation (2), one has

$$f(\mathbf{v}_i) = f_{21}(\mathbf{v}_i,\mathbf{v}_j) = f_{111}(\mathbf{v}_i,\mathbf{v}_j,\mathbf{v}_k) = 0 \quad (1 \le i,j,k \le n). \tag{9}$$

Suppose that $\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{v}_1,\ldots,\mathbf{v}_n$ are linearly dependent over K. In view of the linear independence of $\boldsymbol{\gamma},\mathbf{v}_1,\ldots,\mathbf{v}_n$ over $K(\sqrt{d})$, with either $\mathbf{x} = \boldsymbol{\alpha}$ or with $\mathbf{x} = \boldsymbol{\beta}$ one has that $\mathbf{x},\mathbf{v}_1,\ldots,\mathbf{v}_n$ are linearly independent over K. Moreover f vanishes on the K-space spanned by $\mathbf{x},\mathbf{v}_1,\ldots,\mathbf{v}_n$, since some linear combination of $\boldsymbol{\gamma}$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a non-zero scalar multiple of $\mathbf{x}$. In this case, therefore, the conclusion (4) is immediate.

Thus we may suppose that $\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent over K, and in particular that neither $\boldsymbol{\alpha}$ nor $\boldsymbol{\beta}$ is zero. Suppose that $f(\boldsymbol{\beta}) = 0$. Then the equations (3) are satisfied with $\mathbf{x} = \boldsymbol{\beta}$ so long as $f_{12}(\mathbf{v}_i, \boldsymbol{\beta}) = 0$ $(1 \le i \le n)$. But in such circumstances f vanishes on the K-space spanned by $\boldsymbol{\beta}, \mathbf{v}_1, \dots, \mathbf{v}_n$, so that the desired conclusion again follows in view of the linear independence of $\boldsymbol{\beta}, \mathbf{v}_1, \dots, \mathbf{v}_n$. Suppose then that for some i one has $f_{12}(\mathbf{v}_i, \boldsymbol{\beta}) \ne 0$. Now $\boldsymbol{\alpha} + (\boldsymbol{\beta} + \mathbf{v}_i)\sqrt{d}$ is a solution of the system (3) over $K(\sqrt{d})$, and moreover the latter vector is linearly independent of $\mathbf{v}_1, \dots, \mathbf{v}_n$ over $K(\sqrt{d})$. Further, on recalling (5) and (9), one finds that

$$f(\mathbf{v}_i + \boldsymbol{\beta}) = f(\mathbf{v}_i) + f_{21}(\mathbf{v}_i, \boldsymbol{\beta}) + f_{12}(\mathbf{v}_i, \boldsymbol{\beta}) + f(\boldsymbol{\beta}) = f_{12}(\mathbf{v}_i, \boldsymbol{\beta}) \ne 0.$$

Thus we may suppose without loss of generality that there exists a point $\boldsymbol{\alpha} + \boldsymbol{\beta}\sqrt{d}$ with $\boldsymbol{\alpha}, \boldsymbol{\beta} \in K^s$, such that the set of vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{v}_1, \dots, \mathbf{v}_n$ is linearly independent over K, the equations (5)-(9) hold, $f(\boldsymbol{\alpha} + \boldsymbol{\beta}\sqrt{d}) = 0$ and $f(\boldsymbol{\beta}) \ne 0$.

Define the vectors $\mathbf{w}_0, \dots, \mathbf{w}_n \in K^s$ by

$$\mathbf{w}_0 = \boldsymbol{\alpha} f(\boldsymbol{\beta}) - \boldsymbol{\beta} f_{12}(\boldsymbol{\alpha}, \boldsymbol{\beta}), \quad \mathbf{w}_i = \mathbf{v}_i f(\boldsymbol{\beta}) - \boldsymbol{\beta} f_{12}(\mathbf{v}_i, \boldsymbol{\beta}) \quad (1 \le i \le n).$$

On recalling that $f(\boldsymbol{\beta}) \ne 0$, one finds that the $\mathbf{w}_i$ $(0 \le i \le n)$ are linearly independent over K^s, and by making use of (5)-(9) it may be verified that

$$f(\mathbf{w}_i) = f_{21}(\mathbf{w}_i, \mathbf{w}_j) = f_{111}(\mathbf{w}_i, \mathbf{w}_j, \mathbf{w}_k) = 0 \quad (0 \le i, j, k \le n).$$

Consequently, for each $y_0, \dots, y_n$ the equation (4) is satisfied, and thus the proof of the lemma is complete.

We have now reached a position from which we may launch our proof of Theorem 1. Let K and d satisfy the hypotheses of Theorem 1, and let m be a non-negative integer. Write $L = K(\sqrt{d})$. If γ_L and $\beta_L(m; \gamma_L)$ are not both finite, then there is nothing to prove, so we assume that both γ_L and $\beta_L(m; \gamma_L)$ are finite. When $m = 0$, the conclusion of Theorem 1 is immediate from Lemma 2.1. Suppose then that the conclusion of Theorem 1 holds when $m = n - 1$, where n is some positive integer. If we can establish the conclusion of Theorem 1 with $m = n$, then the theorem will follow by induction.

Let s be an integer with $s > \beta_L(n; \gamma_L) + \frac{1}{2}n(n+3)$, and let $f(\mathbf{x}) \in K[x_1, \dots, x_s]$ be a cubic form. By hypothesis, there exist linearly independent zeros $\mathbf{v}_1, \dots, \mathbf{v}_n \in K^s$ with the property that for each $t_1, \dots, t_n$ the equation (2) holds. We may choose elements $\mathbf{e}_1, \dots, \mathbf{e}_{s-n} \in K^s$ so that the $\mathbf{e}_i$ and $\mathbf{v}_j$ together form a basis for K^s. Let $u_1, \dots, u_{s-n}$ be arbitrary elements of K, and write $\mathbf{x} = u_1\mathbf{e}_1 + \dots + u_{s-n}\mathbf{e}_{s-n}$. On substituting into (3), we obtain a system of homogeneous equations in $\mathbf{u}$, one cubic, n quadratic and (by symmetry) $\frac{1}{2}n(n+1)$ linear, all with K-rational coefficients. If we can find a non-trivial L-rational solution to this system, then this solution will necessarily be linearly independent of $\mathbf{v}_1, \dots, \mathbf{v}_n$, and so the hypotheses of Lemma 2.2 will be satisfied.

In the vector space spanned by the $\mathbf{e}_i$, the system of $\frac{1}{2}n(n+1)$ linear equations vanish on a K-rational subspace of affine dimension $s - n - \frac{1}{2}n(n+1)$. Let $\mathbf{g}_1, \dots, \mathbf{g}_r$ be a basis for the latter subspace, write $\mathbf{x} = y_1\mathbf{g}_1 + \dots + y_r\mathbf{g}_r$, and substitute into (3). We now obtain a system of homogeneous equations, one cubic and n quadratic, with K-rational coefficients, and having r variables. Moreover $r = s - \frac{1}{2}n(n+3) > \beta_L(n; \gamma_L)$, so that the system of quadratic equations necessarily vanish on an L-rational subspace of projective dimension γ_L. Let $\mathbf{h}_0, \dots, \mathbf{h}_{\gamma_L}$ be a basis for the latter subspace, write $\mathbf{x} = z_0\mathbf{h}_0 + \dots + z_{\gamma_L}\mathbf{h}_{\gamma_L}$, and substitute into (3). Now, finally, the system becomes a single cubic equation, with L-rational coefficients, and having more than γ_L variables. It follows that the cubic equation possesses a non-trivial L-rational solution, whence the system (3) possesses a non-trivial L-rational solution linearly independent of $\mathbf{v}_1, \dots, \mathbf{v}_n$. We therefore find that the hypotheses of Lemma 2.2 are satisfied, and so Lemma 2.2 implies that $\gamma_K(1; n) < s$. We have thus established the first conclusion of Theorem 1 with $m = n$, and so the first conclusion of Theorem 1 follows by induction.

Finally, by Leep [10, Corollary 2.4 and Theorem 2.6], one has for each natural number n and non-negative integer k,

$$\beta_L(n; k) \le \tfrac{1}{2}n(n+1)\beta_L(1; 0) + (n+1)k,$$

and thus the second conclusion of Theorem 1 follows immediately from the first.

3. Some consequences of Theorem 1

The proofs of Theorem 2 and Corollary 1 are essentially routine, given the conclusion of Theorem 1 and results available in the literature.

The proof of Theorem 2. (a) Let p be a rational prime, let F be an algebraic extension of $\mathbb{Q}_p$, and let $f(\mathbf{x}) \in F[x_1, \dots, x_s]$ be a cubic form. The coefficients of f all lie in some field extension of $\mathbb{Q}_p$ of finite degree, say $\widetilde{F}$. At this point we could simply work in $\widetilde{F}$, along the lines of Leep and Schmidt [9]. However, in order to abbreviate our exposition we instead apply Theorem 1. There exists an element d in $\widetilde{F}$ with $\sqrt{d} \notin \widetilde{F}$. Write $L = \widetilde{F}(\sqrt{d})$. By Martin [15, Corollary 1.1] one has $\beta_L(n; 0) \le 2n^2 + \eta_n$, where η_n is 0 or 2 according to whether n is even or odd. It therefore follows from Leep [10, Corollary 2.4] that for each natural number n and non-negative integer k, one has $\beta_L(n; k) \le 2n^2 + \eta_n + (n+1)k$. Consequently Theorem 1 implies the bound

$$\gamma_{\widetilde{F}}(1; n) \le 2n^2 + \eta_n + (n+1)\gamma_L + \tfrac{1}{2}n(n+3).$$

Moreover, when $s > \gamma_{\widetilde{F}}(1; n)$, the solution set of the cubic form $f(\mathbf{x})$ contains an F-rational linear subspace of projective dimension n. Part (a) of Theorem 2 follows immediately on making use of the result $\gamma_L = 9$ due to Lewis [11] (see also Demyanov [5] when $p \ne 3$).

(b) Let K be an algebraic extension of $\mathbb{Q}$ and let $f(\mathbf{x}) \in K[x_1, \dots, x_s]$ be a cubic form. Let $\widetilde{K}$ denote the finite field extension of $\mathbb{Q}$ containing the coefficients of $f(\mathbf{x})$. If $\sqrt{-1} \in \widetilde{K}$ then we take d to be any element in $\widetilde{K}$ with $\sqrt{d} \notin \widetilde{K}$. Otherwise we take $d = -1$. Write $L = \widetilde{K}(\sqrt{d})$. In either case we have $\sqrt{-1} \in L$, and so L is a totally complex field. In particular it follows from Hasse [6] that $\beta_L(1; 0) = 4$, and from Pleasants [16] that $\gamma_L \le 15$. Part (b) of Theorem 2 now follows immediately from the final conclusion of Theorem 1.

The proof of Corollary 1 *to Theorem* 2. (a) Let p be a rational prime, let F be an algebraic extension of $\mathbb{Q}_p$, and let $f(\mathbf{x}), g(\mathbf{x}) \in F[x_1, \dots, x_s]$ be cubic forms. Let $\widetilde{F}$ denote the finite field extension of $\mathbb{Q}_p$ containing the coefficients of $f(\mathbf{x})$ and $g(\mathbf{x})$. By Theorem 2(a), the solution set of the equation $f(\mathbf{x}) = 0$ contains an $\widetilde{F}$-rational subspace of projective dimension m provided only that $s > \frac{1}{2}(5m^2 + 21m + \delta_m)$. Let $\mathbf{e}_0, \dots, \mathbf{e}_m$ be a basis for such a subspace, write $\mathbf{x} = y_0\mathbf{e}_0 + \dots + y_m\mathbf{e}_m$, and consider the equation $g(\mathbf{x}) = 0$. In view of the substitution, the latter equation becomes a homogeneous cubic equation in $m + 1$ variables with coefficients in $\widetilde{F}$, and thus by [11] has a non-trivial $\widetilde{F}$-rational solution provided that $m \ge 9$. Thus we deduce that whenever $s > 308$ the system $f(\mathbf{x}) = g(\mathbf{x}) = 0$ possesses a non-trivial F-rational solution, and part (a) of Corollary 1 follows.

(b) The same argument as that used in part (a), mutatis mutandis, shows that when L is an algebraic extension of $\mathbb{Q}$, one has

$$\gamma_L(2) \le \tfrac{1}{2}(5\gamma_L^2 + 37\gamma_L + 30),$$

whence the desired conclusion follows from the bound $\gamma_L \le 15$ of [16].

We conclude by remarking that, subject to non-singularity conditions, stronger bounds are known for the number of variables required to solve systems of cubic equations than have been derived herein (see [3, 7, 8, 23]). However, it seems difficult to exploit such results effectively to obtain corresponding conclusions unconditionally.

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